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Topological Classification of Impulsive Differential Equations

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The present paper is concerned with the topological classification of impulsive differential equations. Under the assumption that the linear part of the right-hand side of the equation considered has an exponential dichotomy and the nonlinear perturbation is small enough, it is proved that for the underlying equations there exist $N + 1$ types topologically different from one another.

1. INTRODUCTION

Impulsive differential equations have been qualitatively investigated by many authors (Bainov *et al.*, 1989a, b; Samoilenko and Perestyuk, 1987; and references therein). Special attention has been focused on the existence of integral manifolds and the dichotomy of solutions (Bainov *et al.*, 1989a, b, n.d.).

The present paper is concerned with the topological classification of impulsive differential equations. To do this we make use of the method developed in Minh (1988) for nonautonomous differential equations. The main difficulty we face is due to the discontinuity of the trajectories of impulsive differential equations. To overcome this we modify the techniques of constructing a homeomorphism using the Morse lemma (see Lemma 4 and Theorem 1 below). As in Minh (1988), introducing the notion of topological equivalence between "proper" integral manifolds, we shall prove that the equations

$$\frac{dx}{dt} = A(t)x + f(t, x) \qquad \text{if} \quad t \neq t_n \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) + h_n(x(t_n)) \tag{2}$$

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are topologically equivalent to the standard equations

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = x_2, \qquad (x_1, x_2) = x$$

under the assumption that the linear part has an exponential dichotomy and the nonlinear perturbation is small enough.

2. STATEMENT OF THE PROBLEM

Suppose $T = \{t_n: n \in \mathbb{Z}\}$ is a sequence of moments in $\mathbb{R}$ satisfying the conditions

$$\text{(i)} \qquad t_n < t_{n+1} \qquad \text{for every} \quad n \in \mathbb{Z}, \qquad \lim_{n \to \pm\infty} t_n = \pm\infty \tag{3}$$

$$\text{(ii)} \qquad \lim_{h \to \infty} \frac{i(t, t+h)}{h} = p < \infty \tag{4}$$

uniformly in $t \in \mathbb{R}$, where $i(a, b)$ denotes the number of moments contained in the interval (a, b).

Consider the impulsive equation

$$\frac{dx}{dt} = A(t)x + f(t, x) \qquad \text{if} \quad t \neq t_n \tag{5}$$

$$x(t_n^+) = Q_n x(t_n) + h_n(x(t_n)) \tag{6}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $A(\cdot)$ is a matrix-valued function, and Q_n is a matrix, under the following assumptions:

(i) $A(\cdot)$ and $f(\cdot, \cdot)$ are extendable to continuous functions on every set of the form $[t_n, t_{n+1}]$ and $[t_n, t_{n+1}] \times \mathbb{R}^N$, respectively.

(ii) Q_n is invertible for every $n \in \mathbb{Z}$.

(iii) $\sup_t \|A(t)\| < \infty$, $\sup_n \|Q_n\| < \infty$, $\sup_t \|f(t, 0)\| < \infty$, and $\sup_n \|h_n(0)\| < \infty$.

(iv) $\|f(t, x) - f(t, y)\| \leq \delta \|x - y\|$ and $\|h_n(x) - h_n(y)\| \leq \delta \|x - y\|$ for all $x, y \in \mathbb{R}^N$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$.

Definition 1. Solution of the impulsive equation (5), (6) we shall call a function satisfying equation (5) for $t \neq t_n$ and equation (6) for $t = t_n$ and being continuous from the left.

Remark. Under the above assumptions the impulsive equation (5), (6) satisfies all conditions of the Existence and Uniqueness Theorem. So we denote by $U(t, s)$ the Cauchy matrix of the homogeneous equation corresponding to (5), (6) and by $X(t, s, x)$ the solution of (5), (6) satisfying $X(s, s, x) = x$ for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

3. PREPARATORY LEMMAS

Before stating the preparatory lemmas we shall use to prove the main result, we need the following notions.

Definition 2. An integral manifold of the impulsive equation (5), (6) is said to be *proper* if $\mathbb{R}^N$ splits up into a direct sum of $\mathbb{R}^k$ and $\mathbb{R}^m$ so that this integral manifold is represented by the equation

$$x_2 = \varphi(t, x_1), \qquad x_1 \in \mathbb{R}_1^k, \quad x_2 \in \mathbb{R}^m \tag{7}$$

where φ is extendable to a continuous mapping on every set of the form $[t_n, t_{n+1}] \times \mathbb{R}^k$; furthermore, $\|\varphi(t, x) - \varphi(t, y)\| \leq \eta \|x - y\|$ for all $x, y \in \mathbb{R}^k$, η is independent of t, and $\varphi(t, 0) = 0$ for every $t \in \mathbb{R}$.

Definition 3. Let M and N be proper integral manifolds of two given impulsive equations. M is said to be *topologically equivalent* to N if there exists a homeomorphism H: $M \to N$ with the following properties:

(i) $H(t, x) = (\theta(t), h_t(x))$, where h_t: $M(t) \to N(\theta(t))$ is a homeomorphism, $M = \{(t, M(t)), t \in \mathbb{R}\}$, $N = \{(t, N(t))\}$, θ: $\mathbb{R} \to \mathbb{R}$ is an orientation-preserving homeomorphism, $\theta(t_n) = \tau_n$ for every $n \in \mathbb{Z}$, and $\{t_n: n \in \mathbb{Z}\}$ and $\{\tau_n: n \in \mathbb{Z}\}$ are moments of the impulse effect of the given equations.

(ii) If $x(t)$ is any solution contained in M, then $h_t(x(t)) = y(\theta(t))$, where $y(\tau)$ is a solution contained in N, and h_t^{-1} has the same property.

(iii) There exists an increasing function L: $[0, \infty) \to [0, \infty)$, $L(0) = 0$, continuous at 0 and such that

$$\|h_t(x)\| \leq L(\|x\|) \qquad \text{for every} \quad (t, x) \in M$$

$$\|h_t^{-1}(y)\| \leq L(\|y\|) \qquad \text{for every} \quad (t, y) \in N$$

Remark 1. (i) The above-defined topological equivalence is an equivalence relation.

(ii) If $\{t_n: n \in \mathbb{Z}\}$ and $\{\tau_n: n \in \mathbb{Z}\}$ satisfy condition (3), (4), then the topological equivalence preserves the boundedness of solutions and the stability of the trivial solution.

(iii) From the condition (iii) imposed on h_t and h_t^{-1} we deduce that

$$\lim_{\|x\| \to \infty} \|h_t(x)\| = \lim_{\|y\| \to \infty} \|h_t^{-1} y\| = \infty \tag{8}$$

From now on we shall deal only with impulsive equations with moments of impulse effect satisfying (3), (4), and integral manifolds satisfying the conditions in Definition 2 except for the condition $\varphi(t, 0) = 0$ for all $t \in \mathbb{R}$.

Definition 4. Let M and N be integral manifolds of two given impulsive equations. M is said to be *topologically weakly equivalent* to N if there exists a homeomorphism H: $M \to N$ satisfying conditions (i) and (ii) in Definition 3 and equality (8).

So if M and N are proper and topologically equivalent, then they are topologically weakly equivalent to each other. It may be noted that weak topological equivalence is an equivalence relation. This preserves the boundedness of solutions.

Definition 5. The homogeneous equation corresponding to (1), (2) is said to have an *exponential dichotomy* if there exist positive constants M and α and a projector P: $\mathbb{R}^N \to \mathbb{R}^N$ such that

$$\|X(t)PX^{-1}(s)\| \leq M \exp(-\alpha(t - s)) \qquad \text{for} \quad t \geq s$$

$$\|X(t)(I - P)X^{-1}(s) \leq M \exp(-\alpha(s - t)) \qquad \text{for} \quad t \leq s$$

where $X(t)$ is a fundamental matrix of the homogeneous equation.

Now consider two linear systems

$$\dot{x} = A(t)x \qquad \text{if} \quad t \neq t_n \tag{9}$$

$$x(t_n^+) = Q_n x(t_n) \tag{10}$$

and

$$\dot{x} = B(t)x \qquad \text{if} \quad t \neq t_n \tag{11}$$

$$x(t_n^+) = R_n x(t_n) \tag{12}$$

Definition 6. Equation (11), (12) is said to be *kinematically similar* to equation (9), (10) if there exists a matrix-valued function $S(\cdot)$ having the following properties:

(i) $S(\cdot)$ is continuous for $t \neq t_n$ and bounded on $\mathbb{R}$.

(ii) $S(t)$ has discontinuities of the first kind at $t = t_n$ and is continuous from the left.

(iii) $S(t)$ is invertible for every $t \in \mathbb{R}$ and $S^{-1}(\cdot)$ enjoys the properties (i), (ii).

(iv) if $x(t)$ is any solution of (11), (12), then $S(t)x(t)$ is a solution of (9), (10).

Lemma 1. Assume that equation (9), (10) has an exponential dichotomy. Then it is kinematically similar to a reducible equation (11), (12), i.e., $B(t)$ and R_n commute with the projector P; in addition, $\|B(t)\| \leq \|A(t)\|$ and $\|R_n\| \leq \|Q_n\|$ for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$.

This lemma is proved by modifying the proof of the well-known result on reducibility (Coppel, 1978; Daleckii and Krein, 1974; Aulbach, 1984; Bainov *et al.*, n.d.).

Lemma 2. Suppose that the homogeneous equation corresponding to (5), (6) has an exponential dichotomy with projector P and positive constants M and α and enjoys the property $f(t, 0) = 0$, $h_n(0) = 0$. Then for sufficiently small δ there exist proper integral manifolds

$$M^- = \{(s, x) \in \mathbb{R} \times \mathbb{R}^N: \|X(t, s, x)\| \to 0 \text{ as } t \to \infty\}$$

$$M^+ = \{(s, x) \in \mathbb{R} \times \mathbb{R}^N: \|X(t, s, x)\| \to 0 \text{ as } t \to -\infty\}$$

Furthermore, if $f(t, \cdot)$ and $h(\cdot)$ are of class C^k, where $k \in \mathbb{N}$, so are these integral manifolds.

For the proof see Samoilenko and Perestyuk (1987). It may be noted that if $f(t, \cdot)$ and $h_n(\cdot)$ are of class C^k, we can show that these integral manifolds are of class C^k, too.

Lemma 3. Suppose all conditions in Lemma 2 are satisfied except for $f(t, 0) = 0$ and $h_n(0) = 0$. Then equation (1), (2) has at least one bounded solution.

For the proof see Samoilenko and Perestyuk (1987).

4. MAIN RESULTS

Lemma 4. Let the homogeneous equation corresponding to (5), (6) have an exponential dichotomy with projector $P = 0$ and positive constants M and α, and in addition, let $f(t, 0) = 0$, $h_n(0) = 0$, and $f(t, \cdot)$, $h_n(\cdot) \in C^3$. Then for δ small enough equation (5), (6) is topologically equivalent to the standard equation

$$\dot{x} = x, \qquad x \in \mathbb{R}^N$$

Proof. The main difficulty in proving this lemma is due to the discontinuity of the trajectories of solutions of equation (5), (6). But the main idea of the proof is suggested by Palmer (1979). First we consider the function

$$V(t, x) = \int_{-\infty}^{t} \|X(\tau, t, x)\|^2 d\tau \tag{13}$$

This function is well defined. In fact, we have the variation of parameters formula

$$X(t, s, x) = U(t, s)x + \int_{s}^{t} U(t, \tau) f(\tau, X(\tau, s, x))\, d\tau \tag{14}$$
$$+ \sum_{s<t_i\leq t} U(t, t_i) Q_i(X(t_i, s, x))$$

Setting $\psi(t) = \|X(t, s, x)\|$, we have for $t \le s$

$$\begin{aligned}\|\psi(t)\| \le{}& M\|x\| \exp(-\alpha(s - t)) \\ &+ \int_t^s M \exp(-\alpha(\tau - t))\, \delta\|\psi(\tau)\|\, d\tau \\ &+ \sum_{t<t_i\le s} M \exp(-\alpha(t_i - t))\, \delta\|\psi(t_i)\|\end{aligned}$$

Thus

$$\begin{aligned}e^{-\alpha t}\|\psi(t)\| \le{}& M\|x\| \exp(-\alpha s) \\ &+ \int_t^s \delta M \exp(-\alpha\tau)\, \|\psi(\tau)\|\, d\tau \\ &+ \sum_{t<t_i\le s} \delta M \exp(-\alpha t_i)\, \|\psi(t_i)\|\end{aligned}$$

Putting $u(t) = e^{-\alpha t}\|\psi(t)\|$ and applying Gronwall's inequality (Samoilenko and Perestuk, 1987; Bainov *et al.*, 1989b), we get

$$u(t) \le \prod_{t<t_i<s} (1 + M\delta)M\|x\| \exp(-\alpha s) \exp(M\delta(s - t))$$

So we have

$$\|\psi(t)\| \le (1 + M\delta)^{i(t,s)}M\|x\| \exp(-(\alpha - M\delta)(s - t)) \tag{15}$$

From this it follows that for δ small enough the integral (13) is absolutely convergent, uniformly with respect to x contained in an arbitrary bounded set of $\mathbb{R}^N$.

From now on we assume that δ is chosen so small that

$$\begin{aligned}&(1 + M\delta)^{i(t,s)}M\|x\| \exp(-(\alpha - M\delta)(s - t)) \\ &\le M\|x\| \exp(-\overline{\alpha}(s - t))\end{aligned} \tag{16}$$

where $\overline{\alpha}$ is a fixed positive constant.

Note that $V(\cdot, x)$ is continuous at $t \ne t_n$ and has a discontinuity of the first kind at $t = t_n$. To prove Lemma 4, we need the following result.

Lemma 5. For δ and ϵ small enough and s fixed, the set $\{x \in \mathbb{R}^N\colon V(s, x) = \epsilon\}$ is homeomorphic to $\{x \in \mathbb{R}^N\colon \|x\| = 1\}$; in addition, if we denote by g_s that homeomorphism, then $g_s(x)$ and $g_s^{-1}(x)$ depend on (s, x) continuously for $s \ne t_n$ and have a discontinuity of the first kind at $s = t_n$.

Proof. We shall make use of the Morse lemma. By calculating $D_x^2V(s, x)|_{x=0}$ we have

$$D_xV(s, x)\big|_{x=0}(\xi) = \int_{-\infty}^{s} 2\langle D_xX(u, s, 0)\xi, X(u, s, 0)\rangle\, du = 0$$

$$\begin{aligned} D_x^2V(s, x)\big|_{x=0}(\xi, \eta) &= 2\int_{-\infty}^{s} \langle D_x^2X(u, s, 0)(\xi, \eta), X(u, s, 0)\rangle\, du \\ &\quad + 2\int_{-\infty}^{s} \langle D_xX(u, s, 0)\xi, D_xX(u, s, 0)\eta\rangle\, du \\ &= 2\int_{-\infty}^{s} \langle D_xX(u, s, 0)\xi, D_xX(u, s, 0)\eta\rangle\, du \end{aligned}$$

We are going to show that

$$\int_{-\infty}^{s} \|D_xX(u, s, 0)\xi\|^2\, du \geq c\|\xi\|^2 \tag{17}$$

for some positive c. In fact, from (14) we deduce that

$$\begin{aligned} D_xX(u, s, 0) &= U(u, s) + \int_{u}^{s} U(u, \tau)D_2f(\tau, X(\tau, s, 0))D_xX(\tau, s, 0)\, d\tau \\ &\quad + \sum_{u<t_i\leq s} U(u, t_i)DQ_i(X(t_i, s, 0))D_xX(u, s, 0) \end{aligned}$$

Without difficulty we can show that

$$\|D_xX(u, s, 0)\| \leq M \exp(-\overline{\alpha}(s - u)) \qquad \text{for } u \leq s \tag{18}$$

Hence

$$\begin{aligned} &\int_{-\infty}^{s} \|D_xX(u, s, 0)\xi\|^2\, du \\ &\quad = \int_{-\infty}^{s} \|U(u, s)\xi\|^2\, du \\ &\quad + \int_{-\infty}^{s} \Bigg\|\Bigg(\int_{u}^{s} U(u, \tau)D_2f(\tau, 0)D_xX(\tau, s, 0)\, d\tau \\ &\quad + \sum_{u<t_i\leq s} U(u, t_i)DQ_i(0)D_xX(u, s, 0)\Bigg)(\xi)\Bigg\|^2\, du \end{aligned}$$

$$+ 2\int_{-\infty}^{s} \left\langle U(u, s)\xi, \left(\left(\int_{u}^{s} U(u, \tau)D_2 f(\tau, 0)D_x X(\tau, s, 0)\, d\tau \right.\right.\right.$$

$$\left.\left.\left. + \sum_{u<t_i\leq s} U(u, t_i)DQ_i(0)D_x X(u, s, 0)\right)(\xi)\right\rangle du$$

Taking into account (18), we can show without difficulty that

$$\int_{-\infty}^{s} \|D_x X(u, s, 0)\xi\|^2\, du \geq \int_{-\infty}^{s} \|U(u, s)\xi\|^2\, du - \delta\overline{M}\|\xi\|^2$$

where $\overline{M}$ is a positive constant independent of ξ.

Since the linear part of (1) has an exponential dichotomy with projector $P = 0$, then (Coppel, 1978) there exists a constant $c_1 > 0$ such that

$$\int_{-\infty}^{s} \|U(u, s)\xi\|^2\, du \geq c_1\|\xi\|^2$$

Finally, if δ is chosen small enough, we get

$$D_x^2 V(s, x)\big|_{x=0}(\xi, \xi) \geq c\|\xi\|^2$$

where c is a positive constant.

Now, making use of the Morse lemma, we see that for ϵ small enough the set $\{x \in \mathbb{R}^N\colon V(s, x) = \epsilon\}$ is homeomorphic to the unit sphere [see Golubitsky and Guillemin (1973) for the details]. In addition, from the proof of the Morse lemma we deduce that $g_s(x)$ depends on (s, x) continuously for $s \neq t_n$ and has a discontinuity at $s = t_n$. This completes the proof of Lemma 5. ■

Now we continue with the proof of Lemma 4. Under the smallness assumption on δ, we have (16) and then for $t > \tau$

$$\|x\| = \|X(\tau, t, X(t, \tau, x))\| \leq M\|X(t, \tau, x)\| \exp(-\overline{\alpha}(t - \tau))$$

Thus

$$M^{-1}\|x\| \exp(\overline{\alpha}(t - \tau)) \leq \|X(t, \tau, x)\| \qquad \text{for} \quad t > \tau \tag{19}$$

Let $x(t)$ denote $X(t, 0, x)$. It is easily shown that $V(t, x(t))$ is continuous on the whole axis. Furthermore, at $t \neq t_n$

$$\frac{d}{dt} V(t, x(t)) = \frac{d}{dt} \int_{-\infty}^{t} \|X(\tau, 0, x)\|^2 \, dx$$
$$= \|X(t, 0, x)\|^2 = \|x(t)\|^2 \tag{20}$$

At $t \neq t_n$ we have

$$D_- V(t_n, x(t_n)) = \lim_{k \to 0} \frac{1}{k} [V(t_n, x(t_n)) - V(t_n - k, x(t_n - k))]$$
$$= \|X(t_n, 0, x)\|^2 = \|x(t_n)\|^2$$

From (16) it follows that

$$V(t, x) \leq \int_{-\infty}^{t} M^2 \|x\|^2 \exp(-2\overline{\alpha}(t - \tau)) \, d\tau$$
$$\leq \frac{M^2}{2\overline{\alpha}} \|x\|^2 \tag{21}$$

Set $K = \sup_t \|A(t)\| + \delta$. For $\tau, t \in (t_n, t_{n+1})$ we have no difficulty in proving that

$$\frac{d}{dt} \ln\|X(t, \tau, x)\| \leq K$$

Thus

$$\|X(t, \tau, x)\| \leq \|x\| \exp(K(t - \tau)) \qquad \text{for} \quad t > \tau$$

Now for $t_k < \tau \leq t_{k+1} < \cdots < t_n < t \leq t_{n+1}$ we have

$$\|X(t, \tau, x)\| \leq L^{i(\tau, t)} \|x\| \exp(K(t - \tau))$$

where $L = \sup_n \|Q_n\| + \delta$. So we can find a positive number K_1 such that

$$\|X(t, \tau, x)\| \leq \|x\| \exp(K_1(t - \tau)) \qquad \text{for} \quad t > \tau \tag{22}$$

For the case when $t < \tau$ we have

$$\|X(\tau, t, X(t, \tau, x))\| \leq \|X(t, \tau, x)\| \exp(K_1(\tau - t))$$

Hence for $t < \tau$

$$\|x\| \exp(K_1(t - \tau)) \leq \|X(t, \tau, x)\| \tag{23}$$

Taking into account (23), we have

$$\begin{aligned} V(t, x) &= \int_{-\infty}^{t} \|X(s, t, x)\|^2 \, ds \\ &\geq \int_{-\infty}^{t} \|x\|^2 \exp(2K_1(s - t)) \, ds \\ &\geq \frac{1}{2K_1} \|x\|^2 \end{aligned} \tag{24}$$

Consider the function $f(t, s, x) = V(t, X(t, s, x))$. For $t \neq t_n$

$$\frac{\partial f}{\partial t}(t, s, x) = \frac{d}{dt} V(t, X(t, s, x)) = \|X(t, s, x)\|^2$$

Combining (21) and (24), we have

$$\frac{M^2}{2\overline{\alpha}} \|X(t, s, x)\|^2 \geq f(t, s, x) \geq \frac{1}{2K_1} \|X(t, s, x)\|^2$$

From (16), (19) it follows that for $x \neq 0$

$$\lim_{t \to -\infty} \|X(t, s, x)\| = 0, \qquad \lim_{t \to \infty} \|(X(t, s, x)\| = \infty$$

Hence

$$\lim_{t \to -\infty} f(t, s, x) = 0, \qquad \lim_{t \to \infty} f(t, s, x) = \infty$$

It is easily shown that the equation $\epsilon = f(t, s, x)$ has a unique solution $t = t(s, x)$, where ϵ is chosen as in Lemma 5; furthermore, t depends on (s, x) continuously when $x \neq 0$.

Now we are in a position to construct the homeomorphism which transforms the underlying equation into the standard one. In fact, we define the homeomorphism $H: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N$ as follows:

$$\begin{cases} h_s(x) = \exp(s - t(s, x)) \, g_{t(s,x)}(X(t(s, x), s, x)) & \text{for} \quad x \neq 0 \\ h_s(0) = 0 \end{cases} \tag{25}$$

where g_s is defined in Lemma 4. We are going to show that H satisfies all properties listed in Definition 3. In fact, for $x \neq 0$, $h_s(x)$ is continuous with respect to (s, x). For $x = 0$

$$\|h_s(x) - h_s(0)\| \leq \exp(s - t(s, x)) \tag{26}$$

We shall estimate the right-hand side of (26). By definition we have

$$V(s, x) - \epsilon = V(s, x) - V(t(s, x), X(t(s, x), s, x)) = \int_{t(s,x)}^{s} D_{-}V(u, X(u, s, x))\, du \tag{27}$$

Hence

$$|s - t(s, x)| \leq |V(s, x) - \epsilon| / \inf_{u \in [t(s,x),s], u \neq t_n} \frac{\partial f}{\partial u}(u, s, x)$$

If $V(s, x) \geq \epsilon$, we have $t(s, x) \leq s$. For $u \in [t(s, x), s]$ we obtain

$$\epsilon \leq f(u, s, x) \leq \frac{M^2}{2\overline{\alpha}} \|X(u, s, x)\|^2 \tag{28}$$

Thus

$$\frac{\partial f}{\partial u}(u, s, x) = \|X(u, s, x)\|^2 \geq \frac{2\epsilon\overline{\alpha}}{M^2}$$

for $u \in [t(s, x), s]$, $u \neq t_n$. So we get

$$0 \leq s - t(s, x) \leq (V(s, x) - \epsilon) \frac{M^2}{2\epsilon\overline{\alpha}} \leq \left(\frac{M^2}{2\overline{\alpha}} \|x\|^2 - 1\right) \frac{M^2}{2\epsilon\overline{\alpha}} \tag{29}$$

Taking into account (24), we see that we have (29) when

$$\|x\| \geq (\epsilon/2K_1)^{1/2}$$

For $\|x\| \leq (2\epsilon\overline{\alpha})^{1/2}/M$ we have $\epsilon \geq V(s, x)$. So from (22), (27) we get

$$0 \leq \epsilon - V(s, x) \leq \|x\|^2 \int_{s}^{t(s,x)} \exp(2K_1(u - s))\, du \leq \frac{\|x\|^2}{2K_1} \{\exp[2K_1(t(s, x) - s)] - 1\}$$

Thus

$$0 \leq \epsilon - \frac{M^2}{2\overline{\alpha}} \|x\|^2 \leq \frac{\|x\|^2}{2K_1} \{\exp[2K_1(t(s, x) - s)] - 1\}$$

$$\frac{1}{2K_1} \ln\left\{\left[\epsilon - \left(\frac{M^2}{2\overline{\alpha}} + \frac{1}{2K_1}\right)\|x\|^2\right] \frac{2K_1}{\|x\|^2}\right\} \leq t(s, x) - s \tag{30}$$

From this it follows that $h_s(x)$ is continuous at $x = 0$. Furthermore, we easily see that there exists an increasing function L: $[0, \infty) \to [0, \infty)$, $L(0) = 0$, continuous at 0, such that

$$\|h_s(x)\| \leq L(\|x\|) \qquad \text{for} \quad x \in \mathbb{R}^N \tag{31}$$

It is easily checked that $H^{-1}: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N$ is defined by the following formula:

$$H^{-1}(s, x) = (s, h_s^{-1}(x))$$

where

$$h_s^{-1}(x) = X(s, u, g_u^{-1}(x/\|x\|)), \qquad u = s - \ln\|x\| \quad \text{for} \quad x \neq 0$$

$$h_s^{-1}(0) = 0$$

It is obvious that for $x \neq 0$, $h_s^{-1}(x)$ is continuous with respect to (s, x). For $\|x\| < 1$ we have $u > s$. So from (16), (24) it follows that

$$\begin{aligned} \|h_s^{-1}(x)\| &\leq M\|g_u^{-1}(x/\|x\|)\| \exp(-\bar{\alpha}(u - s)) \\ &\leq M \frac{\sqrt{\epsilon}}{2K_1} \exp(\bar{\alpha} \ln\|x\|) = M \frac{\sqrt{\epsilon}}{2K_1} \|x\|^{\bar{\alpha}} \end{aligned} \tag{32}$$

For $\|x\| > 1$ we have

$$\begin{aligned} \|h_s^{-1}(x)\| &\leq \|g_u^{-1}(x/\|x\|) \exp(K_1(s - u)) \\ &\leq \frac{\sqrt{\epsilon}}{2K_1} \|x\|^{K_1} \end{aligned} \tag{33}$$

From (32), (33) it is seen that h_s^{-1} is continuous at 0, and there exists an increasing L': $[0, \infty) \to [0, \infty)$, $L'(0) = 0$, continuous at 0, such that

$$\|h_s^{-1}(x)\| \leq L'(\|x\|) \qquad \text{for every} \quad (s, x) \in \mathbb{R} \times \mathbb{R}^N \tag{34}$$

This completes the proof of Lemma 4. ∎

Theorem 1. Let the homogeneous equation corresponding to (5), (6) have an exponential dichotomy with projector P and positive constants M and α, $f(t, 0) = h_n(0) = 0$ for all t, n; let $f(t, \cdot)$ and $h_n(\cdot)$ be of class C^3. Then for δ small enough equation (5), (6) is topologically equivalent to the standard equation

$$\begin{cases} \dot{x}_1 = -x_1, & x_1 \in \mathbb{R}^k \\ \dot{x}_2 = x_2, & x_2 \in \mathbb{R}^m \end{cases}$$

where $k = \dim P(\mathbb{R}^N)$.

Proof. We shall prove this theorem applying the idea of the proof of the Theorem in Minh (1988). First, Q_n and $A(t)$ can be assumed to commute with P. Under the assumptions of the theorem, for every $(s, x) \in \mathbb{R} \times \mathbb{R}^N$ there exist

$$E^-(s, x) = \{y \in \mathbb{R}^N: \|X(t, s, y) - X(t, s, x)\| \to 0, \text{ as } t \to \infty\}$$
$$E^+(s, x) = \{y \in \mathbb{R}^N: \|X(t, s, y) - X(t, s, x)\| \to 0, \text{ as } t \to -\infty\}$$

Suppose

$$\varphi_s^-: E^k = P(\mathbb{R}^N) \to E^m = (I - P)(\mathbb{R}^N)$$
$$\varphi_s^+: E^m \to E^k$$

such that

$$E^-(s, 0) = \{x + \varphi_s^-(x), x \in E^k\}$$
$$E^+(s, 0) = \{\varphi_s^+(x) + x, x \in E^m\}$$

Here, for the sake of convenience, we assume that the scalar product is chosen so that P is an orthogonal projection. We denote

$$E^- = \{(s, \mathrm{E}^-(s, 0)), s \in \mathbb{R}\}, \qquad E^+ = \{(s, E^+(s, 0)), s \in \mathbb{R}\}$$

Then both E^- and E^+ are proper integral manifolds. We shall establish the topological equivalence between E^-, E^+ and E^k, E^m. Suppose $x(t)$ is any solution of (5), (6) contained in E^-. Then $Px(t)$ satisfies

$$\begin{aligned}\frac{d}{dt} Px(t) &= P\frac{dx(t)}{dt} = P^2A(t)x(t) + Pf(t, x(t))\\ &= P^2A(t)x(t) + Pf(t, x(t))\\ &= (PA(t))Px(t) + Pf(t, x(t)) \qquad \text{if} \quad t \neq t_n \qquad (35)\\ Px(t_n^+) &= PQ_nx(t_n) + Ph_n(x(t_n))\\ &= P^2Q_nx(t_n) + Ph_n(x(t_n))\\ &= (PQ_n)Px(t_n) + Ph_n(x(t_n)) \qquad (36)\end{aligned}$$

From Lemma 4 it follows that equation (35), (36), setting $U = Px$, is topologically equivalent to $\dot{x}_1 = -x_1$, $x_1 \in E^k$. Meanwhile, we can easily see that $I \times P$ is a homeomorphism by which E^- is topologically equivalent to equation (35), (36). Finally, we have shown that E^- is topologically equivalent to the standard equation $\dot{x}_1 = -x_1$, $x_1 \in E^k$. Similarly, it is shown that E^+ is topologically equivalent to $\dot{x}_2 = x_2$, $x_2 \in E^m$. We denote by $H^-(t, x) = (t, h_t^-(x))$, $H^+(t, x) = (t, h_t^+(x))$ the homeomorphisms which transform the standard equations $\dot{x}_1 = -x_1$, $\dot{x}_2 = x_2$ into E^-, E^+, respectively. Suppose that $x \in \mathbb{R}^N$, $x = u + v$, $u \in E^k$, and $v \in E^m$. Setting $a = h_t^-(u)$ and $b = h_t^+(v)$, we define H as follows:

$$H(t, x) = (t, h_t(x)) \tag{37}$$

where

$$h_t(x) = E^+(t, a) \cap E^-(t, b)$$

It is shown that the definition of h_t is correct (Tichonova, 1970). It is easy to see that $x = h_t^{-1}(y) = (h_t^-)^{-1}(a) + (h_t^+)^{-1}(b)$, where

$$a = E^+(t, y) \cap E^-(t, 0), \qquad b = E^-(t, y) \cap E^+(t, 0)$$

From the results in Bylov *et al.* (1966) and Tichonova (1970) it follows that h_t and h_t^{-1} are both continuous. Now suppose that $x(t)$ is a solution of the standard system $\dot{u} = -u$, $\dot{v} = v$, $u \in E^k$, $v \in E^{n-k}$, $x = u + v$. Then $h_t(x(t))$ is a solution of (5), (6). In fact, denoting by $Y(t, s, y)$ the solution of

$$\begin{aligned} \frac{dy}{dt} &= A(t)y + f(t, y) \qquad \text{if} \quad t \neq t_n \\ y(t_n^+) &= Q_n y(t_n) + h_n(y(t_n)) \\ y(s) &= y \end{aligned}$$

we have

$$\begin{aligned} Y(t, s, E^-(s, y)) &= E^-(t, Y(t, s, y)) \\ Y(t, s, E^+(s, y)) &= E^+(t, Y(t, s, y)) \end{aligned}$$

Suppose $x(t) = u(t) + v(t)$. We put $a(t) = h_t^-(u(t))$ and $b(t) = h_t^+(v(t))$. So $(t, a(t))$, $(t, b(t))$ belong to E^-, E^+, respectively. We get

$$\begin{aligned} Y(t, 0, a(0)) &= a(t) \\ Y(t, 0, b(0)) &= b(t) \\ Y(t, 0, E^+(0, a(0))) &= E^+(t, a(t)) \\ Y(t, 0, E^-(0, b(0))) &= E^-(t, b(t)) \end{aligned}$$

By definition

$$\begin{aligned} y(t) &= h_t(x(t)) \\ &= E^+(t, a(t)) \cap E^-(t, b(t)) \\ &= Y(t, 0, E^+(0, a(0)) \cap E^-(0, b(0))) \\ &= Y(t, 0, h_0(x(0))) \end{aligned}$$

i.e., $y(t)$ is a solution of (5), (6). It is easy to see that h_t^{-1} has a similar property.

Now we have to prove the existence of a function L with the desired properties. If we choose δ small enough, we get (Bylov *et al.*, 1966; Tichonova, 1970)

$$\|y_2 - b_2\| \leq \frac{1}{2}\|y_1 - b_1\|, \qquad \|y_1 - a_1\| \leq \frac{1}{2}\|y_2 - a_2\|$$

where $y = y_1 + y_2$, $a = h_t^-(u) = a_1 + a_2$, $b = b_t^+(v) = b_1 + b_2$; $y_1, a_1, b_1 \in E^k$; $y_2, a_2, b_2 \in E^{n-k}$. So we get

$$\|y_1\| + \|y_2\| \leq 2(\|a_1\| + \|a_2\| + \|b_1\| + \|b_2\|)$$

Finally, this implies the existence of a function L_1: $[0, \infty) \to [0, \infty)$ with the desired properties such that $\|h_t^{-1}(x)\| \leq L_1(\|x\|)$. In the same way, there exists L_2: $[0, \infty) \to [0, \infty)$. Finally, we choose $L(\|x\|) = \max(L_1(\|x\|), L_2(\|x\|)$ and then we get

$$\sup_{t\in \mathbb{R}} \max(\|h_t(x)\|, \|h_t^{-1}(x)\|) \leq L(\|x\|)$$

This completes the proof of Theorem 1. ∎

Corollary 1. Suppose all conditions of Theorem 1 satisfied except for the condition $f(t, 0) = h_n(0) = 0$. Then equation (5), (6) is topologically weakly equivalent to the standard equation.

Proof. From Lemma 3 it follows that there exists a bounded solution $\bar{x}(t)$ of equation (5), (6). Consider $h_t(x) = \bar{x}(t) + x$. It is easily seen that $H(t, x) = (t, h_t(x))$ is a homeomorphism which transforms equation (5), (6) into another one satisfying all conditions of Theorem 1. This completes the proof of Corollary 1. ∎

Remark 2. In the present paper the finite dimensionality of the phase space of x is needed only to apply the Morse lemma in Lemma 5. We do not know whether this may be omitted as in the case of differential equations without impulse effect.

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